PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Analytical Treatment of Two-Dimensional Supersonic Flow. II. Flow with Weak Shocks

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Phil. Trans. R. Soc. Lond. A 1956 **248**, 499-515
doi: 10.1098/rsta.1956.0007

Analytical treatment of two-dimensional supersonic flow
II. Flow with weak shocks

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(*Communicated by M. J. Lighthill, F.R.S.—Received 2 June 1955*)

A scheme of approximate solution is presented for the treatment of shock waves in the steady, plane flow of a perfect gas. It is based on the neglect of any entropy variations produced by the shocks and hence is applicable only when the shocks are weak. The method provides an extension of Friedrichs's (1948) results for simple waves to wave-interaction regions.

By an examination of the solution of the continuous-flow equations in the neighbourhood of a known shock wave it is shown how the downstream flow may be calculated without reference to the particular shock shape (§2). There are certain cases in which this approach fails and they are discussed by means of a typical example in §3·3. Once the downstream flow has been calculated, it is possible to set up general equations for the determination of the shock (§2). Examples of the solution of these equations for typical problems are given in §3.

In §4 there is a brief discussion of the validity of using homentropic theory and estimates of the errors involved in the solution process are obtained.

1. Introduction

The methods outlined in part I (Mahony & Meyer 1955) provide a means of obtaining the solution of a completely supersonic, irrotational flow field in which there are no shock waves. These methods are here extended to permit an approximate treatment of flows in which weak shocks occur. The flow is then no longer homentropic, and the full problem is one of great complexity for which no analytical, non-degenerate solution is known. However, when the shock waves are sufficiently weak, the changes in entropy are extremely small, and it appears plausible, as a first approximation, to treat the flow as homentropic. On this basis, Friedrichs (1948) and Pillow (1949) have solved the problem of the formation and decay of shock waves in a simple wave.

A similar approach is here used to treat general wave-interaction regions in which weak shocks are present. A qualitative examination of the nature of the flow in the neighbourhood

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of a typical shock shows that, once the homentropic approximation is adopted, the boundary conditions determine a continuous solution free from shocks. The solution is not realistic in that it cannot represent a flow, for it contains regions in which three different velocity vectors are predicted for any one point in the flow plane; but it is mathematically complete and consistent. In the plane of the characteristic invariants, this multivalued nature of the solution does not obtrude and the shock wave appears as a boundary which excludes the unrealistic portion of the continuous solution from the actual flow.

This procedure is reversed in § 3, when a shock is introduced into a known continuous solution, which is not realistic, so that the portion of the solution retained provides a complete, one-valued and physically reasonable flow. This total solution, including a determinate shock, satisfies both the equations of continuous flow and the shock conditions provided that terms of the order of the entropy variations are neglected in all equations. This introduction of a shock wave, whenever the continuous solution breaks down, would appear to be justified by the work of Johannesen (1952), where it is found to yield the correct description of the flow.

While this approach gives a clear idea of the basic method, it is by no means always sufficient for the satisfactory solution of the problem. This is illustrated by the example discussed in § 3·3, in which the shock meets a jet boundary. In this case the calculation of the downstream flow, by using the focusing equations to form an extended solution, is found to give a qualitatively wrong flow. As a result, the problem needs to be treated in two stages. First, an extended flow field is calculated in which neither the shock nor the boundary occurs, but from which both can be determined as far as their point of interaction. Thereafter, a fresh formulation of the problem is required for the calculation of the subsequent flow.

It is not immediately clear that the basic idea of the homentropic approximation is always adequate. Moreover, it has not yet proved possible to compare a solution of the truncated equations with the corresponding solution of the full equations to establish that the two differ only by terms of the same order as the entropy variations. The problem has been investigated by Lighthill (1950) for a specific class of simple wave flows for which the Friedrichs theory seemed most open to doubt. His work, based on a plausible assumption about the flow behind the shock, consists of a detailed examination of the flow, and in particular the energy balance, and yields a consistent and reasonable picture of the rotational flow field. The investigation suggests strongly that the errors, involved in using the homentropic theory to determine the shock, are in fact of the same order as the entropy variations, but that there is a greater error involved in the determination of the structure of the downstream flow. As a result, further shocks cannot be determined to the same accuracy as the first.

The question is considered further in § 4 below, where the general equations governing the flow downstream of a curved shock are obtained in a form which permits easy comparison with a homentropic solution obtained by the methods of part I. For the particular class of problems considered, the leading terms of a solution of the general equations are found by seeking one which differs but little from the homentropic solution. If another solution to the problem exists, it must differ greatly from that obtained using the homentropic approximation. Thus, if the latter provides an approximation at all, an estimate of the errors involved can be obtained by a direct comparison of the present solutions. On this basis it can be shown that the method of approximation introduces errors of third order in the shock strength into the determination of the velocity field but of first order into the calculation of the velocity gradients. This agrees with the results obtained by Lighthill for a different problem by a different method and thus supplies further confirmation of the limits of validity of the homentropic theory. However, for the problem here considered, the shock shape is obtained with a third-order error only if the homentropic approximation is applied by a method, such as is described in § 2, in which larger errors cancel each other.

2. Study of a shock in the hodograph plane

Within the framework of a theory which neglects any small variations of entropy, the use of the characteristic invariants, α and β, as independent variables should simplify the analysis as it has done in part I. Moreover, if a unity of treatment is retained for continuous and discontinuous flows the formulation of a problem does not have to be changed if a satisfactory continuous flow is found not to exist. Thus it is first necessary to express the Rankine–Hugoniot shock-jump conditions in a form suited to the hodograph plane* rather than in the more usual forms which are used in the flow plane.

As this treatment is based upon the neglect of the entropy variations produced by any shock which may occur, it is satisfactory if any other equally small quantities are also neglected and then the appropriate form of the shock relations can be obtained as follows. The changes in the flow variables across a weak shock differ from those across a simple wave, producing the same stream deflexion from the same initial state, by terms which are of third order in the shock strength (Courant & Friedrichs 1948). Let the shock strength δ be defined as the stream deflexion

$$\delta = \theta_2 - \theta_1, \tag{1}$$

where henceforth the suffixes 1 and 2 denote respectively the conditions on the upstream and downstream sides of the shock. Since a simple wave is characterized by having one of the characteristic invariants constant across it, either $(\alpha_2-\alpha_1)$ or $(\beta_2-\beta_1)$ must be $O(\delta^3)$ which is also the order of the entropy change across the shock, and it follows from (I, 3)† that the difference (t_2-t_1) must be negative. Then from (I, 1, 2) and (1) it follows that if $\delta > 0$,

$$\left.\begin{aligned} &\alpha_2-\alpha_1 = O(\delta^3), \quad \beta_2-\beta_1 = 2\delta+O(\delta^3), \quad t_2-t_1 = -\delta+O(\delta^3),\\ \text{and if } \delta<0, \quad &\alpha_2-\alpha_1 = 2\delta+O(\delta^3), \quad \beta_2-\beta_1 = O(\delta^3), \quad t_2-t_1 = \delta+O(\delta^3). \end{aligned}\right\} \tag{2}$$

It is only necessary to consider one of these cases because the treatment of the other is closely analogous, and so it will be assumed that δ is positive. The other expression for the change in a flow variable, which is required for later work, is the formula (Howarth 1953)

$$\mu_2-\mu_1 = (2N_1-1)\,\delta+2N_1(2N_1-1)\tan\mu_1\,\delta^2+O(\delta^3), \tag{3}$$

where

$$N = \frac{1}{2}\left(1-\frac{\mathrm{d}\mu}{\mathrm{d}t}\right) = \frac{\gamma+1}{4}\sec^2\mu, \tag{4}$$

* I.e. the (α, β)-plane. Note that, provided only that the flow is homenergic, the variables α and β are defined as functions of θ and t and hence the (α, β)-plane is obtained by a known fixed transformation of the conventional hodograph plane.

† References given in the form (I, n) refer to part I, equation number n.

for the change in Mach angle. The specification of the shock in the flow plane is completed by the relation*

$$\omega = \mu_1 + N_1\delta + \tfrac{1}{2}N_1^2 \cot\mu_1[1+3\tan^2\mu_1 - 8\sin^2\mu_1/(\gamma+1)]\,\delta^2 + O(\delta^3), \tag{5}$$

which determines the acute angle, ω, which the shock makes with the upstream flow direction.

These relations will now be used to discuss the correspondence between the flow and hodograph planes in the neighbourhood of a shock wave. A simple case is treated to establish the general features of the correspondence, but it is easy to extend the treatment to more complicated problems. The example chosen is that of a known flow with a shock wave which starts with zero strength at some interior point of the flow field. It is assumed that there are no other shock waves or singularities of the wave-front or branch-line type in the neighbourhood of the shock. For definiteness, it is also assumed that β is increasing in the flow direction and that α is increasing in the direction of growth of the shock.

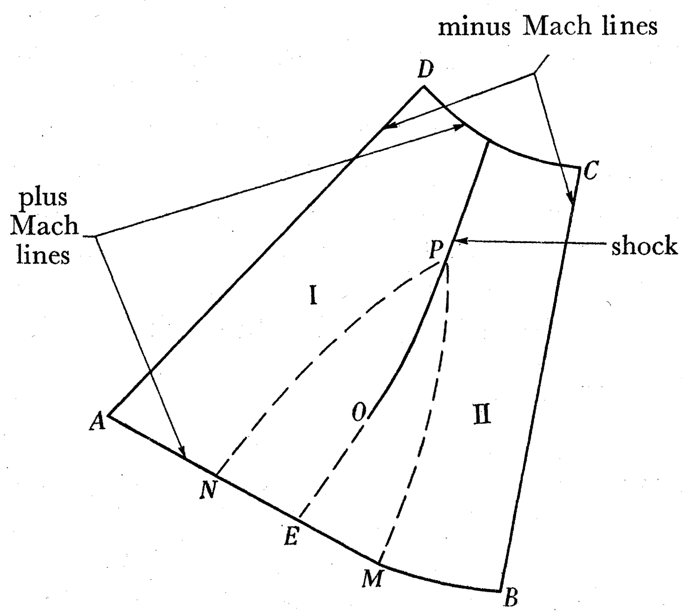

FIGURE 1*a*. Flow plane.

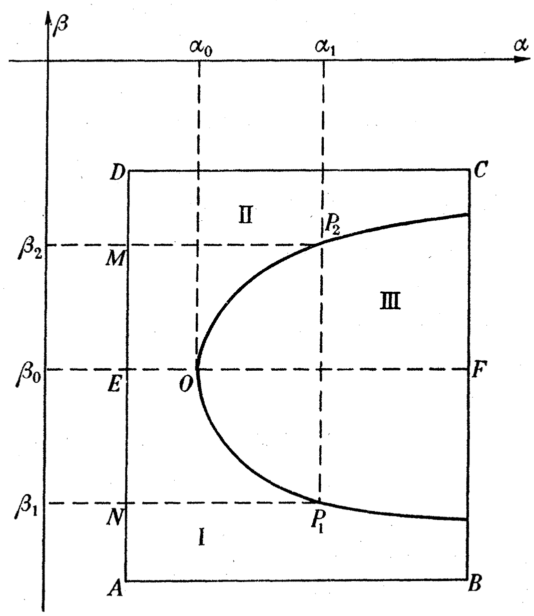

FIGURE 1*b*. Hodograph plane.

Then to each point on the shock in the flow plane there corresponds two points P_1, (α_1, β_1), and P_2, (α_2, β_2), in the hodograph plane. There is one exception to this; the birth point of the shock in the flow plane has only one corresponding point (α_0, β_0) in the hodograph plane. Since, in the flow plane, the shock approximately bisects the angle between the upstream and downstream minus Mach lines, the above assumptions imply that

$$\alpha_1, \alpha_2 > \alpha_0,$$

$$\beta_1 < \beta_0 < \beta_2,$$

and hence the correspondence is as depicted in figure 1 *a*, *b*. Note that region III, which is enclosed by the shock wave in the hodograph plane, corresponds to no part of the actual flow.

* This can be deduced by series substitution in the formula relating ω and μ_1 given in Howarth (1953).

The focusing equations (I, 9, 10) may be used to continue the known solutions in regions I and II into region III. For, the sides AB, AD and CD of the characteristic rectangle $ABCD$ are completely in the known flow, and hence h_α is known on AB and CD while h_β is known on AD. These values of the characteristic length parameters on AB and AE uniquely determine a continuous flow solution in the rectangle $ABFE$ and the values on CD and DE similarly determine a solution in $CDEF$. From the continuity properties of the characteristic length parameters established by Meyer (1949) and the assumed continuity in regions I and II, it follows that these two solutions are continuous across EOF. By inverting the procedure in rectangle $CDEF$ and using the data on DE and that calculated on EOF from the rectangle $ABFE$, it is possible to determine the solution in $CDEF$. Thus the solution in region II, which is the actual downstream flow, may be calculated from the data on AB and AD by ignoring the existence of the shock and using the uniquely determined extended continuous solution.

It is of interest to examine the properties of the extended continuous solution in region III. For convenience, all quantities which are $O(\delta^3)$ will be placed equal to zero in this paragraph, but the results obtained here may be confirmed by the general theory developed later in this section. The points P_1 and P_2 lie on the same plus Mach line and the length P_1P_2 is thus $\int_{\beta_1}^{\beta_2} h_\beta \, d\beta$. Furthermore, this length must be zero for P_1 and P_2 correspond to the same point in the flow plane. By the initial assumptions h_β is continuous in regions I and II and hence is continuous in region III,* so that h_β must vanish somewhere on each plus Mach line in region III. Moreover, as regions I and II in the hodograph plane correspond to a flow free of mapping singularities, h_β is of the same sign throughout these regions and in particular at P_1 and P_2. Therefore, h_β must vanish an even number of times on the segment of each plus Mach line within region III, or, what is equivalent, each plus Mach line encounters an even number of limit lines of the other family. At the point O, in particular,

$$h_\beta = \frac{\partial h_\beta}{\partial \beta} = 0,$$

and the first non-vanishing derivative of h_β with respect to β is of even order.

The transformation relations (I, 5, 6) may be used to define a correspondence between region III and a set of points in the flow plane, or, rather, an extension of this plane. From the behaviour of the solution in the hodograph plane and the properties of limit lines (Meyer 1949), it can be shown that the extended flow plane consists of a folded surface in the neighbourhood of O. The lines of folding of the surface correspond to the limit lines which are cusped at O, while the actual flow is carried on the two extreme sheets with the shock appearing as a cut from one extreme sheet to the other.

The extension of the solution of the focusing equations in the hodograph plane, corresponding to an actual flow with a shock wave, has thus been shown to possess the typical features associated with the failure of a continuous flow solution. This result provides confirmatory justification for the idea, used by many authors in simple-wave theories, of eliminating limit lines by introducing shock waves. If the use of a homentropic theory is

* This follows from lemma I of Meyer (1949) or from the continuity properties of solutions of equations (I, 14, 15).

justified, the above considerations show that such a shock must start from the cusp of the limit line. Furthermore, the downstream flow will remain unaltered by the introduction of a shock wave.

There now remains only the question of the actual determination of the shock shape. The information about the shock, which has not yet been used, is the fact that the loci of the points P_1 and P_2 map on to a single line, in the flow plane whose slope is a given function of the shock strength. That their map is a single line implies by (I, 5, 6) that, for each α_1,

$$0 = x(\alpha_2,\beta_2)-x(\alpha_1,\beta_1) = \int_{\beta_1}^{\beta_2}[h_\beta \cos(\theta-\mu)]_{\alpha=\alpha_1}\,d\beta+\int_{\alpha_1}^{\alpha_2}[h_\alpha \cos(\theta+\mu)]_{\beta=\beta_2}\,d\alpha$$

and $$0 = y(\alpha_2,\beta_2)-y(\alpha_1,\beta_1) = \int_{\beta_1}^{\beta_2}[h_\beta \sin(\theta-\mu)]_{\alpha=\alpha_1}\,d\beta+\int_{\alpha_1}^{\alpha_2}[h_\alpha \sin(\theta+\mu)]_{\beta=\beta_2}\,d\alpha.$$

If the first of these equations is multiplied by $\sin(\theta_2+\mu_2)$ and the second by $\cos(\theta_2+\mu_2)$, the difference of the resulting equations yields

$$\int_{\beta_1}^{\beta_2}[h_\beta \sin(\theta_2+\mu_2+\mu-\theta)]_{\alpha=\alpha_1}\,d\beta = -\int_{\alpha_1}^{\alpha_2}[h_\alpha \sin(\theta_2-\theta+\mu_2-\mu)]_{\beta=\beta_2}\,d\alpha,$$
$$= O(h_\alpha\delta^6), \tag{6}$$

since both $[\theta_2-\theta]_{\beta=\beta_2}$ and $[\mu_2-\mu]_{\beta=\beta_2}$ are $O(\alpha_2-\alpha)$ and $(\alpha_2-\alpha_1)$ is $O(\delta^3)$. The condition that this single line in the flow plane makes an angle ω with the upstream flow direction may be shown, by the help of (I, 5, 6), to imply that the slope of the upstream branch of the shock in the hodograph plane is given by

$$\frac{d\beta_1}{d\alpha_1} = -\frac{h_\alpha(\alpha_1,\beta_1)}{h_\beta(\alpha_1,\beta_1)}\frac{\sin(\omega-\mu_1)}{\sin(\omega+\mu_1)}. \tag{7}$$

A similar formula could be found for the downstream branch, but this is implied by equations (6) and (7). Not only is the present form convenient but also later work, on the effect of the entropy variations, shows that it is necessary if accuracy is not to be lost in certain cases. If appropriate use is made of the shock-jump conditions, equations (6) and (7) serve to determine any two of α_1, β_1 and δ as functions of the third. Thus the shock may be determined in the hodograph plane and this may be transformed to determine the shock in the flow plane.

3. Examples of the solution method

The form of the solution of equations (6) and (7) for the determination of the shock shape is governed by the properties of the extended solution of the focusing equations. Thus different problems require somewhat different treatment, depending upon the nature of the mapping singularity with which the shock is associated, the appropriate form for the early approximations for the continuous solution, and the number and type of singularities, such as wave fronts or branch lines, which occur in the immediate neighbourhood of the shock. The examples chosen illustrate the method appropriate to each of the three possible types of singularity from which a shock may start. In the first example (§ 3·1) the shock starts from two edges of regression; in the second (§ 3·2) it starts from the cusp of a limit line, while in the third (§ 3·3) the birth-point of the shock is the junction of an edge of regression and a limit line. The effect of the method by which the continuous solution has been obtained is illustrated by the first two examples. In § 3·1 the continuous solution has been obtained by the 'inconsistent rule' while in § 3·2 it has been obtained by the method of power-series expansions. The last example illustrates the procedure when a further singularity complicates the calculation of the shock.

3·1. *Thin aerofoil in a non-uniform stream*

The first problem considered is that of calculating the shock attached to the leading edge of a thin sharp-nosed aerofoil in a slowly diverging nozzle. It will be assumed that the small stream deflexions produced by the nozzle in the absence of the aerofoil, and vice versa, are of the same order of magnitude. Attention will be confined to the flow over the upper surface

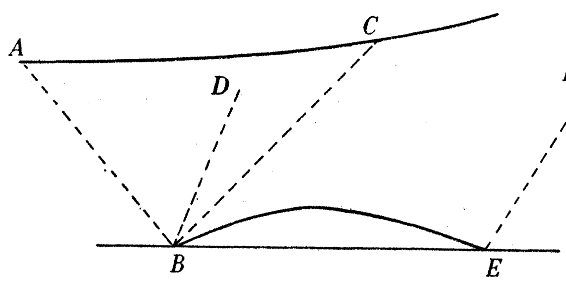

FIGURE 2*a*. Flow plane.

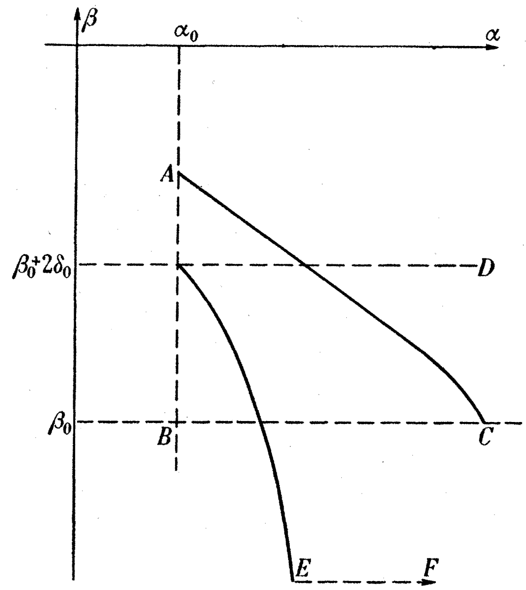

FIGURE 2*b*. Hodograph plane.

of the aerofoil and then the correspondence is as represented diagrammatically in figure 2*a*, *b*. The solutions of the focusing equations in the various regions of the hodograph plane may then be obtained by applying the 'inconsistent rule' of part I. If the aerofoil chord is taken as the unit of length, the characteristic length parameters are $O(\delta_0^{-1})$, where δ_0 is the small angle between the upper surface of the aerofoil and the incident stream at the leading edge. Thus, it can be shown that, in the region bounded by the plus Mach line AB, the minus Mach line BC and the streamline AC (region I),

$$\left.\begin{aligned} V &= V_0(\beta) - \int_{\alpha_0}^{\alpha} m_0\, U_0(\alpha)\, \mathrm{d}\alpha + O(\delta_0) \\ \text{and} \qquad U &= U_0(\alpha) + \int_{\Phi(\alpha)}^{\beta} m_0 V_0(\beta)\, \mathrm{d}\beta + O(\delta_0), \end{aligned}\right\} \tag{8}$$

where (α_0, β_0) is the upstream point B in the hodograph plane, $U_0(\alpha)$ and $V_0(\beta)$ are functions determined by the nozzle geometry alone, and $\beta = \Phi(\alpha)$ is the equation, in the hodograph plane, of the wall streamline. In the region between the two minus Mach lines BC and DB' and the plus Mach line BB' (region II) the solution is

$$\left.\begin{aligned} U &= U_0(\alpha) + \int_{\Phi(\alpha)}^{\beta_0} m_0 V_0(\beta)\,\mathrm{d}\beta + O(\delta_0), \\ \text{and} \quad V &= -\int_{\alpha_0}^{\alpha} m_0 U_0(\alpha)\,\mathrm{d}\alpha + O(\delta_0), \end{aligned}\right\} \tag{9}$$

since the segment BB' maps on to the one point B in the flow plane. Finally, in the region bounded by the minus Mach lines $B'D$ and EF and the aerofoil $B'E$ (region III), the solution is

$$\left.\begin{aligned} U &= U_0(\alpha) + \int_{\Phi(\alpha)}^{\beta_0} m_0 V_0(\beta)\,\mathrm{d}\beta + \int_{\beta_0+2\delta_8}^{\beta} m_0 V_1(\beta)\,\mathrm{d}\beta + O(\delta_0) \\ \text{and} \quad V &= V_1(\beta) - \int_{\Psi(\beta)}^{\alpha} m_0 U_0(\alpha) + O(\delta_0), \end{aligned}\right\} \tag{10}$$

where $V_1(\beta)$ is a determinable function of the aerofoil and nozzle shapes and $\alpha = \Psi(\beta)$ is the map of the aerofoil in the hodograph plane.

As consideration is being limited to the case where only one shock wave occurs, $U\mathrm{d}\alpha$ and $V\mathrm{d}\beta$ are positive definite in regions I and III, equation (9) implies that $U\mathrm{d}\alpha$ is positive in region II while V is negative there. Since V is also negative in regions I and III, the mapping on to the flow plane will be multivalued if, and only if, β is increasing along a plus Mach line in region II; that is, if δ_0 is positive. If δ_0 is negative the solution of the focusing equations provides the generalization of a Prandtl–Meyer fan for non-uniform flow. Thus only the case δ_0 positive will be considered here.

The first step in the determination of the shock wave is the estimation of the orders of magnitude of the quantities involved. Thus the expressions (8), (9) and (10) for the characteristic length parameters are substituted into equations (6) and (7) for the shock shape and only the leading terms are retained. Then the equations

$$\left.\begin{aligned} \frac{\mathrm{d}\beta_1}{\mathrm{d}\alpha_1} &\sim \frac{U_0(\alpha_1)}{V_0(\beta_1)} m_0 \delta \\ \text{and} \quad \int_{\beta_1}^{\beta_0} V_0(\beta)\,\mathrm{d}\beta - 2m_0\delta_0 \int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,\mathrm{d}\alpha &+ \int_{\beta_0+2\delta_0}^{\beta_2} V_1(\beta)\,\mathrm{d}\beta \sim 0 \end{aligned}\right\} \tag{11}$$

are obtained. From the first of these equations it follows that $(\beta_0 - \beta_1)$ is $O(\delta_0^2)$, and this in conjunction with the second equation implies $(\beta_2 - \beta_0 - 2\delta_0)$ is $O(\delta_0^2)$ and hence $\delta - \delta_0$ is $O(\delta_0^2)$ from equation (2). Having established these results, it is possible to rewrite equations (11) in the form

$$\left.\begin{aligned} \frac{\mathrm{d}\beta_1}{\mathrm{d}\alpha_1} &= -m_0\delta_0 \frac{U_0(\alpha_1)}{V_0(\beta_1)} \{1 + O(\delta_0)\} \\ \text{and} \quad \int_{\beta_1}^{\beta_0} V_0(\beta)\,\mathrm{d}\beta + \int_{\beta_0+2\delta_0}^{\beta_2} V_1(\beta)\,\mathrm{d}\beta &= 2m_0\delta_0 \int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,\mathrm{d}\alpha \{1 + O(\delta_0)\}. \end{aligned}\right\} \tag{12}$$

If the notation

$$\left.\begin{aligned} \delta_0 Y(\beta_1) &= \int_{\beta_1}^{\beta_0} V_0(\beta)\,\mathrm{d}\beta, \\ \delta_0 Z(\beta_2) &= \int_{\beta_0+2\delta_0}^{\beta_2} V_1(\beta)\,\mathrm{d}\beta \\ \text{and} \quad X(\alpha_1) &= \int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,\mathrm{d}\alpha, \end{aligned}\right\} \tag{13}$$

where X, Y and Z all range from zero to order unity, is introduced, equations (12) may be written in the simplified form

$$\frac{dY}{dX} = m_0\{1+O(\delta_0)\}$$

and

$$Y+Z = 2m_0 X_0\{1+O(\delta_0)\}.$$

Thus the solution of equations (12) is

$$\left.\begin{aligned} Y &= m_0 X\{1+O(\delta_0)\} = Z,\\ \text{or}\qquad \int_{\beta_1}^{\beta_0} V_0(\beta)\,d\beta &= m_0\delta_0\int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,d\alpha\{1+O(\delta_0)\} = \int_{\beta_0+2\delta_0}^{\beta_2} V_1(\beta)\,d\beta,\end{aligned}\right\} \tag{14}$$

which is a pair of equations determining β_1 and β_2, and hence the shock strength δ, as functions of α_1.

These equations suffice for numerical purposes, but it is possible to obtain an explicit analytical solution under fairly general conditions. It may be recalled from the discussion of the basis of the 'inconsistent rule' of part I, that the boundary conditions, which in essence determine the functions U_0, V_0 and V_1 are applied in the form $x = X(\theta/\delta_0)$. For most aerofoils and nozzles the derivatives of θ/δ_0 with respect to x will be of unit magnitude at most and then the nth order derivatives of U_0, V_0 and V_1 are $O(\delta_0^{-n-1})$. If V_0 and V_1 are expanded in Taylor series about β_0 and $(\beta_0+2\delta_0)$ respectively, and use is made of the orders of magnitude of $(\beta_0-\beta_1)$ and $(\beta_2-\beta_0-2\delta_0)$, the two extreme integrals in equations (14) can be shown to be approximated by the leading terms of their series expansions. Thus the shock shape, in the hodograph plane, is given by

$$\left.\begin{aligned} \beta_0-\beta_1 &= \frac{m_0\delta_0}{V_0(\beta_0)}\int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,d\alpha\{1+O(\delta_0)\},\\ \beta_2-(\beta_0+2\delta_0) &= \frac{m_0\delta_0}{V_1(\beta_0+2\delta_0)}\int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,d\alpha\{1+O(\delta_0)\},\\ \text{and}\qquad \delta-\delta_0 &= \tfrac{1}{2}m_0\delta_0\left\{\frac{1}{V_0(\beta_0)}+\frac{1}{V_1(\beta_0+2\delta_0)}\right\}\int_{\alpha_0}^{\alpha_1} U_0(\alpha)\,d\alpha\{1+O(\delta_0)\}.\end{aligned}\right\} \tag{15}$$

The error terms in the solution are of the same order as the entropy terms which have already been neglected and hence equations (14) or (15) provide the solution sought. In deriving this result only the leading terms of the expressions (8) and (10) have been used and this might be taken to imply that there is no need to continue the solution of the focusing equations through region II. This is indeed so for the example treated, but it is not difficult to see that, by suitable choice of the orders and form of U_0, V_0 and V_1, one could construct a reasonable example in which this continuation would be essential. Thus the extra terms have been included in the present simple example to indicate the way to extend the method in more complicated problems.

3·2. *Shock starting from the cusp of a limit line*

In this example it will be assumed that a solution of the focusing equations has been obtained by the method of power-series expansions, and this solution possesses a cusped limit line which is an envelope of minus Mach lines. If (α_0, β_0) is the point in the hodograph plane at which this limit line

$$h_\beta(\alpha,\beta) = 0$$

is cusped, it follows that
$$h_\beta(\alpha_0,\beta_0)=\left[\frac{\partial h_\beta}{\partial\beta}\right]_{\alpha_0,\beta_0}=0.$$

and hence the shape of the limit line in the hodograph plane is approximately parabolic. Since the branches of the shock wave must lie outside this limit line, it follows that $(\alpha_1-\alpha_0)$ is $O\{(\beta_0-\beta_1)^2\}$ and $(\alpha_2-\alpha_0)$ is $O\{(\beta_2-\beta_0)^2\}$ at most. If this limit line is thus to be replaced by a shock, the shock wave must produce a positive deflexion (see equation (2)) and hence $(\beta_2-\beta_1)$ must be positive. However, the continuous solution already defines a stream direction in the hodograph plane and this may conflict with the requirement that β_2 is greater than β_1. By an examination of the properties of this anomalous case in the flow plane, it can be shown that it arises from flow over a boundary on a folded surface and hence cannot correspond to a real flow and so will not be considered here. Thus the problem will be restricted to the case where β is increasing in the direction of the stream which implies that, on both sides of the shock, h_β is positive.

If the problem is such as to permit the use of the double power series method to obtain the early approximations to the continuous solution, the characteristic length parameters may be expanded in Taylor series using equations (I, 9, 10). The shock must lie outside the limit line in the hodograph plane and so it can be shown that

$$h_\alpha=h_\alpha(\alpha_0,\beta_0)\{1-N_0\cot 2\mu_0(\beta-\beta_0)+O(\delta_0^2)\} \tag{16}$$

and
$$h_\beta=h_\alpha(\alpha_0,\beta_0)\{a(\beta-\beta_0)^2-m_0(\alpha-\alpha_0)+b(\beta-\beta_0)^3+c(\alpha-\alpha_0)(\beta-\beta_0)+O(\delta^4)\}, \tag{17}$$

where
$$a=\left[\frac{1}{2}\frac{\partial^2 h_\beta}{\partial\beta^2}\Big/h_\alpha\right]_{\alpha_0,\beta_0}>0,$$

$$b=\left[\frac{1}{6}\frac{\partial^3 h_\beta}{\partial\beta^3}\Big/h_\alpha\right]_{\alpha_0,\beta_0}$$

and
$$c=\left[m^2\cos 2\mu+\frac{1}{2}\frac{\mathrm{d}m}{\mathrm{d}t}\right]_{\alpha_0,\beta_0}$$

are all known constants in a specific problem. Equations (6) and (7) may be solved by substituting in them equations (16) and (17) and the trial expansions

$$\alpha_1-\alpha_0=A_0(\beta_1-\beta_0)^2+A_1(\beta_1-\beta_0)^3+O(\delta^4) \tag{18}$$

and
$$\delta=D_0(\beta_1-\beta_0)+D_1(\beta_1-\beta_0)^2+O(\delta^3). \tag{19}$$

The largest order terms yield the two equations

$$m_0A_0(2D_0-1)=-a$$

and
$$m_0A_0=a(1+2D_0+\tfrac{4}{3}D_0^2),$$

whence, by elimination of A_0, the cubic

$$D_0^2(D_0+1)=0$$

may be obtained. The three real roots of this equation correspond to cuts made from any sheet of the folded flow plane to any other. The required solution is the one corresponding to a cut between the two extreme sheets, and hence the remaining coefficients in the trial substitutions may then be shown to be

$$A_0 = a/(3m_0),$$

$$D_1 = -\tfrac{1}{10}\{c/m_0 + 3b/a + 2(N_0-1)\cot 2\mu_0 - 2m_0[(2N_0-1)\tan\mu_0 - \cot\mu_0 - \tfrac{1}{2}\cot\mu_0\{1+3\tan^2\mu_0 - 8\sin^2\mu_0/(\gamma+1)\}]\},$$

and $$A_1 = ca/(3m_0^2) + b/m_0 + \tfrac{2}{3}a[(2N_0-1)\tan\mu_0 - \cot\mu_0 - \tfrac{1}{2}\cot\mu_0\{1+3\tan^2\mu_0 - 8\sin^2\mu_0/(\gamma+1)\}].$$

At this stage it is as well to note that there are certain inconsistencies in the method of approximation used in this example. Although h_β has been calculated by neglecting the third-order entropy terms, it has been necessary to retain the third-order terms in h_β itself in order to determine the second-order terms in the shock strength. Furthermore, while the third-order terms are determined by the present solution for α_1 they are not determined for α_2. Thus if equation (7) was replaced by an equivalent equation for $d\beta_2/d\alpha_2$ the position would be reversed. This question is treated in detail in § 4, where it is shown that the above procedure does in fact lead to the correct solution but the other possible methods do not.

3·3. *First shock in an expanding supersonic jet*

In the two previous examples the use of the complete solution of the focusing equations to obtain the downstream flow has proved quite satisfactory. However, its indiscriminate use may sometimes lead to a qualitatively wrong description of the flow. As an illustration the determination of the shock wave occurring near the end of the first period of a jet is now discussed. In §§ 3·1·1 and 3·1·2 of part I, it was shown that, for sufficiently small δ_0,* a limit line occurs in the simple wave region XI, and this limit line starts from the leading plus Mach line provided the initial Mach number exceeds a certain value. Attention is confined to this case, as the geometry is then a little less complicated.

The basic problem is that of calculating the shock originating from the junction of a limit line and an edge of regression. But it is complicated by the awkward nature of the correspondence between the flow and hodograph planes, and so it is instructive to examine the correspondence in detail. In the folded flow plane (figure 3*a*) the upstream sheet contains a uniform flow bounded by the extreme streamline *CD* and the wave front *ABC* of which the segment *BC* is an edge of regression. The middle sheet is bounded by this edge of regression, the limit line *BE* and the streamline *CE*, while the downstream sheet is bounded by the streamline *EF*, the limit line *BE* and the wave front *AB*. The shock appears as a cut *BG* joining the two extreme sheets. The mapping into the hodograph plane is degenerate due to the fact that portions of the flow are uniform or simple waves. Where the flow is of simple wave nature it is convenient to work in terms of s and β (part I, § 3·1·1) instead of α and β. Thus two separate correspondences must be considered, in the first of which the development of the shock in the simple wave is treated. In the (s, β)-plane (figure 3*b*) the whole of the upstream sheet of the flow plane is mapped on to the line

* Here, in contrast to part I, δ_0 is used for the stream deflexion produced at the lip of the jet to avoid confusion with the shock strength δ.

$\beta = \beta_0 - 2\delta_0$, but the remainder of the mapping of the simple wave into the characteristic plane is regular.

Quite analogously to the derivation of equation (7) it can be shown that the equation governing the growth of the shock is

$$\frac{\mathrm{d}\beta_2}{\mathrm{d}s} = -\frac{f(\mu_N)}{f(\mu)\,V}\frac{\sin(\omega-\mu_2-\delta)}{\sin(\omega+\mu_2-\delta)}.$$

Thus from the value of V_{XI} (I, 67), it can be shown that the equation to the shock is

$$s = 1 - 4\psi_0\delta_0/m_0 + 16\psi_0\delta/(3m_0) + O(\delta_0^2).$$

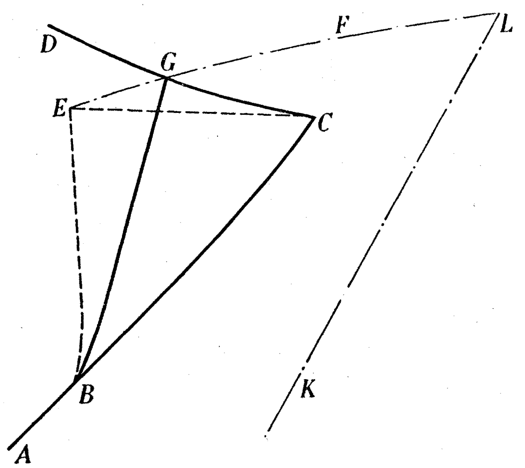

FIGURE 3*a*. Flow plane.

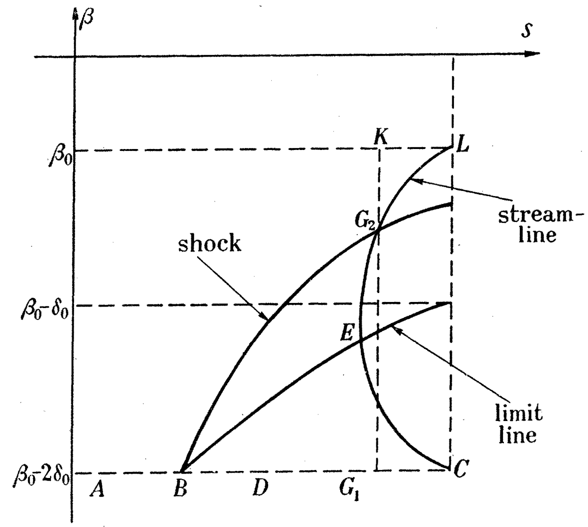

FIGURE 3*b*. Characteristic plane.

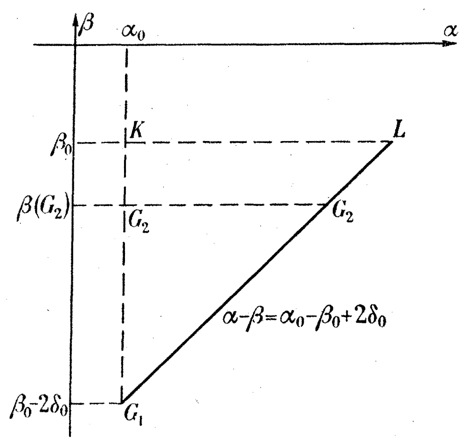

FIGURE 3*c*. Hodograph plane.

The point G is the point of intersection of this line with the streamline

$$\frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{f(\mu_N)}{f(\mu)\,V}$$

through D. Hence its co-ordinates may be shown to be

$$\beta(G) = \beta_0 - \tfrac{1}{2}\delta_0 + O(\delta_0^2)$$

and

$$s(G) = 1 - \tfrac{3}{2}\psi_0\delta_0^2 + O(\delta_0^3).$$

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The second part of the problem, involving the interaction of the shock and the simple wave with the constant-pressure streamline, may now be treated. Since this portion of the flow will be a region of interaction of simple waves, it is convenient to return to the use of α and β as independent variables. The map of the flow in the hodograph plane is represented diagrammatically in figure 3*c*. Thus the new formulation of the boundary conditions is

(1) on the constant pressure streamline, $\alpha-\beta=\alpha_0-\beta_0+2\delta_0$,

$$U=V, \tag{20}$$

(2) on the plus Mach line through G, $\alpha=\alpha_0$,

$$V\equiv 0 \quad (\beta_0-2\delta_0\leqslant\beta<\beta(G_2)), \tag{21}$$

$$V=V_{\mathrm{XI}}(\beta, s(G)) \quad (\beta(G_2)<\beta\leqslant\beta_0). \tag{22}$$

Note that there are two other seemingly possible formulations, but these will be shown to be incorrect. The first possibility is to apply the condition of constant pressure on the portion CEG_2 of the streamline CF. As there is a variation in β of order δ_0 along this segment there would be a corresponding variation of α producing a wave downstream before the shock meets the boundary. In this case it would be impossible to satisfy the shock jump condition that $(\alpha_2-\alpha_1)$ is $O(\delta^3)$, since the upstream flow is uniform. The second incorrect procedure is to apply the value of V_{XI} derived from the extended continuous solution in place of equation (21). If this is done it is easy to show that the solution of the focusing equations and the amended boundary conditions contains a limit line $U=0$ which would give rise to an 'expansion shock', and so this formulation must also be rejected.

The formulation described above leads to no such difficulties and yields a physically consistent picture of the flow. The boundary conditions (20), (21) and (22) determine a unique solution of the focusing equations in the triangle G_1KL, and the solution in the sub-triangle bounded by $\alpha=\alpha_0$, $\beta=\beta(G_2)$ and the constant-pressure streamline is obviously $U\equiv V\equiv 0$. Thus equation (21) may be replaced by

$$U=0 \quad \text{on} \quad \beta=\beta(G_2) \quad (\alpha_0\leqslant\alpha\leqslant\alpha_0-\beta_0+2\delta_0+\beta(G_2)).$$

It is easy to show that this solution yields a correspondence with the flow plane, free of singularities. Moreover, the reflexion of the shock from the constant-pressure streamline is a centred expansion wave and hence is in accord with physical experience.

4. The effect of the entropy variations

As has already been noted in § 3·2 there is considerable doubt as to the accuracy of the homentropic theory applied to the formation of a shock from the cusp of a limit line. Moreover, this case is not covered by Lighthill's investigation (1950), and so this case will be discussed here by a comparison of the homentropic solution with the leading terms of a solution of the full equations. The equations governing a rotational flow field will be formulated in such a way as to facilitate this comparison. Thus, although they are no longer characteristic invariants, α and β will be retained as independent variables, while a set of four correction functions X_α, X_β, Y_α and Y_β will be used as the dependent variables. They are defined by the mapping relations

$$\mathrm{d}x=(h_\alpha+X_\alpha)\cos(\theta+\mu)\,\mathrm{d}\alpha+(h_\beta+X_\beta)\cos(\theta-\mu)\,\mathrm{d}\beta \tag{23}$$

and

$$\mathrm{d}y=(h_\alpha+X_\alpha+Y_\alpha)\sin(\theta+\mu)\,\mathrm{d}\alpha+(h_\beta+X_\beta+Y_\beta)\sin(\theta-\mu)\,\mathrm{d}\beta \tag{24}$$

between the hodograph and flow planes. If the flow were homentropic, but not necessarily equal to the approximate solution, both Y_α and Y_β would vanish identically. Thus, although the effects are interrelated, X_α and X_β may be interpreted as being rough measures of the effect of applying approximate boundary conditions on the shock, while Y_α and Y_β measure the distortion of the flow due to entropy gradients.

Under suitable assumptions as to the existence and continuity of derivatives,

$$\frac{\partial}{\partial\beta}\{X_\alpha \cos(\theta+\mu)\} = \frac{\partial}{\partial\alpha}\{X_\beta \cos(\theta-\mu)\}$$

and
$$\frac{\partial}{\partial\beta}\{(X_\alpha+Y_\alpha)\sin(\theta+\mu)\} = \frac{\partial}{\partial\alpha}\{(X_\beta+Y_\beta)\sin(\theta-\mu)\},$$

since, from (I, 5, 6), h_α and h_β satisfy similar equations. Thus, by a process analogous to the derivation of the focusing equations, it can be shown that

$$\frac{\partial X_\alpha}{\partial\beta}+m(X_\alpha\cos 2\mu - X_\beta) = \operatorname{cosec} 2\mu \cos(\theta-\mu)\left[\frac{\partial}{\partial\alpha}\{Y_\beta\sin(\theta-\mu)\}-\frac{\partial}{\partial\beta}\{Y_\alpha\sin(\theta+\mu)\}\right] \quad (25)$$

and

$$\frac{\partial X_\beta}{\partial\alpha}+m(X_\alpha - X_\beta\cos 2\mu) = \operatorname{cosec} 2\mu \cos(\theta+\mu)\left[\frac{\partial}{\partial\alpha}\{Y_\beta\sin(\theta-\mu)\}-\frac{\partial}{\partial\beta}\{Y_\alpha\sin(\theta+\mu)\}\right]. \quad (26)$$

These equations are purely the result of the formal definitions of the correction functions and are valid regardless of whether or not the solution represents a flow. It will represent a flow if the correspondence satisfies the equations of motion

$$\mathrm{d}\alpha+\sin 2\mu\,\mathrm{d}\Phi = 0 \quad \text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan(\theta-\mu),$$

$$\mathrm{d}\beta-\sin 2\mu\,\mathrm{d}\Phi = 0 \quad \text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan(\theta+\mu)$$

and
$$\mathrm{d}\Phi = 0 \quad \text{on} \quad \frac{\mathrm{d}y}{\mathrm{d}x} = \tan\theta,$$

where
$$\Phi = S/\{2\gamma(\gamma-1)\,C_v\},$$

and S is the specific entropy and C_v the specific heat at constant volume of the gas. In terms of α and β as independent variables these equations may be written in the form

$$1+\sin 2\mu\left[\frac{\partial\Phi}{\partial\alpha}+\tan\phi_1\frac{\partial\Phi}{\partial\beta}\right] = 0, \quad (27)$$

$$1-\sin 2\mu\left[\frac{\partial\Phi}{\partial\alpha}\cot\phi_2+\frac{\partial\Phi}{\partial\beta}\right] = 0, \quad (28)$$

$$\frac{\partial\Phi}{\partial\alpha}+\tan\phi_3\frac{\partial\Phi}{\partial\beta} = 0, \quad (29)$$

where ϕ_1, ϕ_2 and ϕ_3 are the local slopes in the hodograph plane of the plus Mach line, the minus Mach line and the streamline. These three equations for $\partial\Phi/\partial\alpha$ and $\partial\Phi/\partial\beta$ are consistent if, and only if,

$$1-\tan\phi_3\cot\phi_2 = \tan\phi_3-\tan\phi_1. \quad (30)$$

From equations (23), (24) and (30) and the definitions of ϕ_1, ϕ_2 and ϕ_3, it follows that

$$2(h_\alpha+X_\alpha)(h_\beta+X_\beta)\sin 2\mu\{Y_\beta \operatorname{cosec} 2(\theta+\mu)+Y_\alpha \operatorname{cosec} 2(\theta-\mu)\}$$
$$=(h_\alpha+X_\alpha)Y_\beta^2\sin(\theta-\mu)\operatorname{cosec}(\theta+\mu)-(h_\beta+X_\beta)Y_\alpha^2\sin(\theta+\mu)\operatorname{cosec}(\theta-\mu)+Y_\alpha Y_\beta\{(h_\alpha+X_\alpha)$$
$$-(h_\beta+X_\beta)+2\cos\theta\cos\mu[(h_\beta+X_\beta)\sec(\theta+\mu)-(h_\alpha+X_\alpha)\sec(\theta-\mu)]\}. \quad (31)$$

The remaining equation needed to determine X_α, X_β, Y_α and Y_β can be obtained from equation (29) and the entropy distribution on the shock wave. However, as this is not known until the flow is determined the equation cannot be written down in a convenient explicit form.

Then, in principle, the correction functions may be determined from the entropy distribution equation and equations (25), (26) and (31) subject to the boundary conditions appropriate to a given problem. Thus upstream of the shock the flow is homentropic and independent of the shock shape, so that the correction functions vanish identically there. The upstream branch of the shock is thus governed by the same equation,

$$\frac{d\beta_1}{d\alpha_1}=-\frac{h_\alpha(\alpha_1,\beta_1)}{h_\beta(\alpha_1,\beta_1)}\frac{\sin(\omega-\mu_1)}{\sin(\omega+\mu_1)}, \quad (32)$$

as in the homentropic theory. Moreover, in the region downstream bounded by the streamline and the plus Mach line through the birth-point of the shock the flow is homentropic, and hence Y_α and Y_β vanish there. By continuity X_β must vanish on this plus Mach line and $(X_\alpha+Y_\alpha)$ and $(X_\beta+Y_\beta)$ must be continuous across this streamline. The final boundary condition needed is that on the downstream side of the shock which may be expressed in either of two forms, both of which will be useful in later work. The first, which is analogous to equation (32), is

$$\frac{d\beta_2}{d\alpha_2}=-\frac{(h_\alpha+X_\alpha)\sin(\omega-\mu_2-\delta)+Y_\alpha\sin(\theta_2+\mu_2)\cos(\theta_1+\omega)}{(h_\beta+X_\beta)\sin(\omega+\mu_2-\delta)+Y_\beta\sin(\theta_2-\mu_2)\cos(\theta_1+\omega)}, \quad (33)$$

the differential equation for the downstream branch of the shock. The second can be obtained by integrating the mapping relations (23) and (24) along a contour which lies always in the real flow and making use of the fact that (α_1,β_1) and (α_2,β_2) correspond to the same point in the flow plane. Thus it follows that

$$\int_{\alpha_0}^{\alpha_1}[h_\alpha\cos(\theta+\mu)]_{\beta=\beta_1}d\alpha=\int_{\beta_1}^{\beta_2}[h_\beta\cos(\theta-\mu)]_{\alpha=\alpha_0}d\beta+\int_{\alpha_0}^{\alpha_2}[(h_\alpha+X_\alpha)\cos(\theta+\mu)]_{\beta=\beta_2}d\alpha \quad (34)$$

and

$$\int_{\alpha_0}^{\alpha_1}[h_\alpha\sin(\theta+\mu)]_{\beta=\beta_1}d\alpha=\int_{\beta_1}^{\beta_2}[h_\beta\sin(\theta-\mu)]_{\alpha=\alpha_0}d\beta+\int_{\alpha_0}^{\alpha_2}[(h_\alpha+X_\alpha+Y_\alpha)\sin(\theta+\mu)]_{\beta=\beta_2}d\alpha. \quad (35)$$

In attempting to solve this problem a solution will be sought in which $|X_\alpha/h_\alpha|$, $|X_\beta/h_\beta|$ $|Y_\alpha/h_\alpha|$ and $|Y_\beta/h_\beta|$ are all small compared with unity—that is, a solution in which the homentropic theory gives a reasonable approximation to the velocity gradients. The leading terms of such a solution will be found, and this will permit an estimate of the errors involved in a homentropic theory. When these ratios are small, it follows from equation (31) that

$$Y_\beta\operatorname{cosec}2(\theta+\mu)+Y_\alpha\operatorname{cosec}2(\theta-\mu)=o(Y_\alpha),$$

and so
$$Y_\alpha + o(Y_\alpha) = -\sin 2(\theta-\mu)\operatorname{cosec} 2(\theta+\mu)\, Y_\beta$$
$$= o(h_\beta). \tag{36}$$

Thus the leading order terms of equations (32) and (33) are just those of the homentropic theory, and so the leading term for the shock of the solution sought will be the same as given by the homentropic theory. This establishes the orders of magnitude of the variations of all flow quantities, and these can now be used to estimate the contribution of any term.

To simplify the analysis the origin in the hodograph plane will be shifted to the birth-point of the shock. With the established orders of magnitude it is possible to show that the equation to the streamlines is

$$\alpha - \alpha_2 = -\{m_0\alpha_2\beta - \tfrac{1}{6}a\beta^3\}\{1+o(1)\},$$

and, as the entropy is convected along the streamlines and has the value $\lambda\alpha_2^{\frac{3}{2}}$ on the shock,* the entropy distribution is given by

$$\Phi(\alpha,\beta) = \lambda\{\alpha - \beta(\tfrac{1}{6}a\beta^2 - m_0\alpha_2)\}^{\frac{3}{2}}\{1+o(1)\}.$$

From equation (36) and the relations between Y_α, Y_β and the entropy gradients it follows that

$$Y_\alpha = \text{const.}\, h_\beta\{\alpha - \beta(\tfrac{1}{6}a\beta^2 - m_0\alpha_2)\}^{\frac{1}{2}}\{1+o(1)\}$$
and
$$Y_\beta = \text{const.}\, h_\beta\{\alpha - \beta(\tfrac{1}{6}a\beta^2 - m_0\alpha_2)\}^{\frac{1}{2}}\{1+o(1)\},$$

so that both vanish on the streamline through the origin. Hence X_α and X_β are continuous across this line and the equations (25) and (26) yield the solution

$$X_\beta = \cos(\theta+\mu)\sin(\theta-\mu)\operatorname{cosec} 2\mu\, Y_\beta + O(h_\alpha\alpha^2)$$
and
$$X_\alpha = \cos(\theta-\mu)\sin(\theta-\mu)\operatorname{cosec} 2\mu\, Y_\beta/h_\beta + \text{const.}\, h_\alpha\alpha^{\frac{1}{2}} + O(h_\alpha\alpha),$$

where the constant in the last equation could be determined from equations (34) and (35) and the third-order shock conditions.

Thus it has proved possible to find the leading terms of a solution which does satisfy all the necessary equations as well as the condition that the four ratios $|X_\alpha/h_\alpha|$, $|X_\beta/h_\beta|$, $|Y_\alpha/h_\alpha|$ and $|Y_\beta/h_\beta|$ are all small. It will be assumed that this is the appropriate physical solution to the problem, and it is now possible to use the solution obtained for the correction functions to estimate the errors involved in a homentropic theory. It is immediately apparent that the use of h_α instead of $(h_\alpha + X_\alpha)$ downstream will involve a comparatively large proportional error of $O(\delta)$. As such terms would be required to compute the second-order terms in any succeeding shock, it follows that any downstream shock can only be determined to the first order by a homentropic theory. If equations (34) and (35) are combined in the same way as in the derivation of equation (6), then it can be shown that the correction functions contribute at most terms $O(h_\alpha\delta^5)$. As such terms do not influence the second-order terms in the determination of the shock shape by homentropic theory, the terms so determined must agree with the solution of the full equations. This is so only because the largest order error, that involving X_α, cancels during the determination. Moreover, for this solution it can be shown that at any point in the flow field the errors in the velocity field as predicted by the simple solution is of the same order as the entropy variations. Note that this approach

* Where λ is a parameter determined by the shock jump conditions and the geometry of the shock.

achieves similar results as obtained by Lighthill by a different method for a somewhat different problem. It would therefore seem probable that the use of the homentropic solution will be valid for a wide range of problems.

In conclusion, the author would like to express his gratitude to Professor M. J. Lighthill, F.R.S., for his encouragement and to Dr R. E. Meyer for his helpful criticism during the preparation of this paper. The author is also indebted to the Australian Department of Supply for a generous grant which permitted him to study at Manchester University.

References

Courant, R. & Friedrichs, K. O. 1948 *Supersonic flow and shock waves*. New York: Interscience.
Friedrichs, K. O. 1948 *Commun. Pure Appl. Math.* **1**, 211.
Howarth, L. (editor) 1953 *Modern developments in fluid dynamics: high speed flow*, **1**. Oxford: Clarendon Press.
Johannesen, N. H. 1952 *Phil. Mag.* **43**, 568.
Lighthill, M. J. 1950 *Phil. Mag.* **41**, 1101.
Mahony, J. J. & Meyer, R. E. 1955 *Phil. Trans.* A, **248**, 467 (part I).
Meyer, R. E. 1949 *Phil. Trans.* A, **242**, 153.
Pillow, A. F. 1949 *Proc. Camb. Phil. Soc.* **45**, 558.

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